Continuity & Differentiability

1. Which of the following statements is true for the function (2024)

$$f(x) = \begin{cases} x^2 + 3, & x \neq 0 \\ 1 \quad , & x = 0 \end{cases} ?$$

(A) f(x) is continuous and differentiable $\forall\, x \in \mathbb{R}$

(B) f(x) is continuous $\forall\, x \in \mathbb{R}$

(C) f(x) is continuous and differentiable $\forall\, x \in \mathbb{R} - \{0\}$

(D) f(x) is discontinuous at infinitely many points

Ans. (C) f(x) is continuous and differentiable $\forall\, x \in \mathbb{R} - \{0\}$

2. Let f(x) be a continuous function on [a, b] and differentiable on (a, b). Then, this function f(x) is strictly increasing in (a, b) if (2024)

(A) $f'(x) < 0, \forall\, x \in (a, b)$

(B) $f'(x) > 0, \forall\, x \in (a, b)$

(C) $f'(x) = 0, \forall\, x \in (a, b)$

(D) $f(x) > 0, \forall\, x \in (a, b)$

Ans. (B) $f'(x) > 0, \forall\, x \in (a, b)$

3. Check whether the function $f(x) = x^2 |x|$ is differentiable at x = 0 or not. (2024)

Ans.

$$f(x) = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x \leq 0 \end{cases}$$

$$RHD = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} h^2 = 0$$

$$LHD = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} (-h^2) = 0$$

$\because$ ***RHD = LHD = 0, So f(x) is differentiable at x*** $= 0$

4.

If $y = \sqrt{\tan \sqrt{x}}$, prove that $\sqrt{x}\, \dfrac{dy}{dx} = \dfrac{1 + y^4}{4y}$.

(2024)

Ans.

$$y = \sqrt{tan\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{sec^2\sqrt{x}}{2\sqrt{tan\sqrt{x}}} \times \frac{1}{2\sqrt{x}}$$

$$\sqrt{x}\frac{dy}{dx} = \frac{sec^2\sqrt{x}}{4\sqrt{tan\sqrt{x}}}$$

$$= \frac{1+(tan\sqrt{x})^2}{4\sqrt{tan\sqrt{x}}} = \frac{1+y^4}{4y}$$

Previous Years' CBSE Board Questions

5.2 Continuity

MCQ

1. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is continuous at
(a) $x = 1$ (b) $x = 1.5$ (c) $x = -2$ (d) $x = 4$
(2023)

2. If the function $f(x) = \begin{cases} 3x-8, & \text{if } x \le 5 \\ 2k, & \text{if } x > 5 \end{cases}$ is continuous, then the value of k is
(a) 2/7 (b) 7/2 (c) 3/7 (d) 4/7
(Term I, 2021-22) Ap

3. The function $f(x) = [x]$, where $[x]$ is the greatest integer function that is less than or equal to x, is continuous at
(a) 4 (b) −2 (c) 1.5 (d) 1
(Term I, 2021-22) U

VSA (1 mark)

4. The value of λ so that the function f defined by $f(x) = \begin{cases} \lambda x, \text{if } x \le \pi \\ \cos x, \text{if } x > \pi \end{cases}$ is continuous at $x = \pi$ is ________. *(2020)* Ap

5. Determine the value of the constant 'k' so that the function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \ge 0 \end{cases}$ is continuous at $x = 0$. *(Delhi 2017)* Ap

6. Determine the value of 'k' for which the following function is continuous at $x = 3$.
$$f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x-3}, & x \ne 3 \\ k, & x = 3 \end{cases}$$
(AI 2017) Ap

SA I (2 marks)

7. Find the value(s) of 'λ', if the function $f(x) = \begin{cases} \frac{\sin^2 \lambda x}{x^2}, \text{if } x \ne 0 \\ 1, \text{if } x = 0 \end{cases}$ is continuous at $x = 0$. *(2023)*

8. Find the relationship between a and b so that the function f defined by $f(x) = \begin{cases} ax+1 \text{ if } x \le 3 \\ bx+3 \text{ if } x > 3 \end{cases}$ is continuous at $x = 3$. *(2021C)* Ap

LA I (4 marks)

9. Find the values of p and q, for which
$$f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \pi/2 \\ p, & \text{if } x = \pi/2 \\ \frac{q(1-\sin x)}{(\pi - 2x)^2}, & \text{if } x > \pi/2 \end{cases}$$
is continuous at $x = \pi/2$. *(Delhi 2016)* Ap

10. Find the value of the constant k so that the function f, defined below, is continuous at $x = 0$, where
$$f(x) = \begin{cases} \left(\frac{1-\cos 4x}{8x^2}\right), & \text{if } x \ne 0 \\ k, & \text{if } x = 0 \end{cases}$$
(AI 2014C) Ap

5.3 Differentiability

MCQ

11. The function $f(x) = |x|$ is
(a) continuous and differentiable everywhere.
(b) continuous and differentiable nowhere.
(c) continuous everywhere, but differentiable everywhere except at $x = 0$.
(d) continuous everywhere, but differentiable nowhere.
(2023)

12. The derivative of x^{2x} w.r.t. x is
(a) x^{2x-1} (b) $2x^{2x}\log x$
(c) $2x^{2x}(1 + \log x)$ (d) $2x^{2x}(1 - \log x)$ *(2023)*

13. If $y^2(2 - x) = x^3$, then $\left(\frac{dy}{dx}\right)_{(1,1)}$ is equal to
(a) 2 (b) −2 (c) 3 (d) −3/2
(Term I, 2021-22) Ap

14. The function $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 2-x & \text{for } x \ge 1 \end{cases}$ is
(a) not differentiable at $x = 1$
(b) differentiable at $x = 1$
(c) not continuous at $x = 1$
(d) neither continuous nor differentiable at $x = 1$
(Term I, 2021-22) Ev

15. If $\sec^{-1}\left(\frac{1+x}{1-y}\right) = a$, then $\frac{dy}{dx}$ is equal to
(a) $\frac{x-1}{y-1}$ (b) $\frac{x-1}{y+1}$
(c) $\frac{y-1}{x+1}$ (d) $\frac{y+1}{x-1}$ *(2020C)* Ap

VSA (1 mark)

16. If $y = \tan^{-1} x + \cot^{-1} x$, $x \in R$, then $\frac{dy}{dx}$ is equal to ________. *(2020)* Ev

17. If $\cos(xy) = k$, where k is a constant and $xy \ne n\pi$, $n \in Z$, then $\frac{dy}{dx}$ is equal to ________. *(2020)* Ev

18. Differentiate $\sin^2(\sqrt{x})$ with respect to x. *(2020)* Ap

19. Let $f(x) = x|x|$, for all $x \in R$ check its differentiability at $x = 0$. *(2020)* Ev

20. If $y = f(x^2)$ and $f'(x) = e^{\sqrt{x}}$, then find $\frac{dy}{dx}$. *(2020)* Ev

21. If $f(x)=x+1$, find $\frac{d}{dx}(fof)(x)$. *(Delhi 2019)* U

SA I (2 marks)

22. If $(x^2+y^2)^2=xy$, then find $\frac{dy}{dx}$. *(2023)*

23. If $f(x)=\begin{cases}x^2, \text{if } x\ge 1\\ x, \text{if } x<1\end{cases}$, then show that f is not differentiable at $x=1$. *(2023)*

24. Check the differentiability of $f(x)=|x-3|$ at $x=3$. *(2021C)* Ev

25. If $y=\sqrt{a+\sqrt{a+x}}$, then find $\frac{dy}{dx}$. *(2020C)* Ap

26. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x. *(2018)* Ev

27. Find $\frac{dy}{dx}$ at $x=1, y=\frac{\pi}{4}$ if $\sin^2 y+\cos xy=K$. *(Delhi 2017)* Ev

LA I (4 marks)

Here, question 28(i) to (iii) is a case study based question of 4 marks.

28. Let $f(x)$ be a real valued function. Then its
    - Left Hand Derivative (L.H.D.): $Lf'(a)=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$
    - Right Hand Derivative (R.H.D.): $Rf'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

    Also, a function $f(x)$ is said to be differentiable at $x=a$ if its L.H.D. and R.H.D. at $x=a$ exist and both are equal.

    For the function $f(x)=\begin{cases}|x-3|, x\ge 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, x<1\end{cases}$

    answer the following questions:

    (i) What is R.H.D. of $f(x)$ at $x=1$?

    (ii) What is L.H.D. of $f(x)$ at $x=1$?

    (iii) Check if the function $f(x)$ is differentiable at $x=1$.

    OR

    (iii) Find $f'(2)$ and $f'(-1)$. *(2023)*

29. Find the values of a and b, if the function f defined by $f(x)=\begin{cases}x^2+3x+a, & x\le 1\\ bx+2, & x>1\end{cases}$ is differentiable at $x=1$. *(Foreign 2016)* Ap

30. If $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2\le 1$ then find $\frac{dy}{dx}$. *(NCERT Exemplar, Delhi 2015)* Ap

31. If $f(x)=\sqrt{x^2+1}; g(x)=\frac{x+1}{x^2+1}$ and $h(x)=2x-3$, then find $f'[h'(g'(x))]$. *(AI 2015)* Ap

32. Show that the function $f(x)=|x-1|+|x+1|$, for all $x\in R$, is not differentiable at the points $x=-1$ and $x=1$. *(AI 2015)* Ev

33. Find whether the following function is differentiable at $x=1$ and $x=2$ or not.
$$f(x)=\begin{cases}x, & x<1\\ 2-x, & 1\le x\le 2\\ -2+3x-x^2, & x>2\end{cases}$$ *(Foreign 2015)* Ap

34. For what value of λ the function defined by $f(x)=\begin{cases}\lambda(x^2+2), & \text{if } x\le 0\\ 4x+6, & \text{if } x>0\end{cases}$ is continuous at $x=0$? Hence check the differentiability of $f(x)$ at $x=0$. *(AI 2015C)* Ap

35. If $\cos y = x\cos(a+y)$, where $\cos a\ne \pm 1$, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$. *(Foreign 2014)* Ev

36. If $y=\sin^{-1}\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\}$ and $0<x<1$, then find $\frac{dy}{dx}$. *(AI 2014C)* Ap

5.4 Exponential and Logarithmic Functions

MCQ

37. If $y=\log(\sin e^x)$, then $\frac{dy}{dx}$ is
    (a) $\cot e^x$ (b) $\operatorname{cosec} e^x$
    (c) $e^x\cot e^x$ (d) $e^x \operatorname{cosec} e^x$ *(2023)*

38. If $y=\tan^{-1}(e^{2x})$, then $\frac{dy}{dx}$ is equal to
    (a) $\frac{2e^{2x}}{1+e^{4x}}$ (b) $\frac{1}{1+e^{4x}}$
    (c) $\frac{2}{e^{2x}+e^{-2x}}$ (d) $\frac{1}{e^{2x}-e^{-2x}}$ *(Term I, 2021-22)* Ev

VSA (1 mark)

39. If $y=\log(\cos e^x)$, then find $\frac{dy}{dx}$. *(NCERT, AI 2019)* Ap

LA I (4 marks)

40. If $y=e^{x^2\cos x}+(\cos x)^x$, then find $\frac{dy}{dx}$. *(2020)* Ev

41. If $\log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx}=\frac{x+y}{x-y}$. *(Delhi 2019)* Ev

42. If $y=\frac{x\cos^{-1}x}{\sqrt{1-x^2}}-\log\sqrt{1-x^2}$, then prove that $\frac{dy}{dx}=\frac{\cos^{-1}x}{(1-x^2)^{3/2}}$. *(Delhi 2015C)* Ap

43. If $e^x+e^y=e^{x+y}$, prove that $\frac{dy}{dx}+e^{y-x}=0$ *(Foreign 2014)* Ev

44. If $y=\tan^{-1}\left(\frac{a}{x}\right)+\log\sqrt{\frac{x-a}{x+a}}$, prove that $\frac{dy}{dx}=\frac{2a^3}{x^4-a^4}$. *(AI 2014C)* Ap

5.5 Logarithmic Differentiation

SA II (3 marks)

45. If $e^{y-x}=y^x$, prove that $\frac{dy}{dx}=\frac{y(1+\log y)}{x\log y}$. *(2021C)* Ev

LA I (4 marks)

46. If $y=x^3(\cos x)^x+\sin^{-1}\sqrt{x}$, find $\frac{dy}{dx}$. *(2020)* Ev

47. If $y=(\log x)^x+x^{\log x}$, then find $\frac{dy}{dx}$. *(NCERT, 2020)* Ap

48. Find $\frac{dy}{dx}$, if $x^y\cdot y^x=x^x$. *(2019C)* Ap

49. If $x^y-y^x=a^b$, find $\frac{dy}{dx}$. *(Delhi 2019)* Ev

50. If $y=(x)^{\cos x}+(\cos x)^{\sin x}$, then find $\frac{dy}{dx}$. *(AI 2019)*

51. Differentiate the function $(\sin x)^x+\sin^{-1}\sqrt{x}$ with respect to x. *(Delhi 2017)* Ap

OR

If $y=(\sin x)^x+\sin^{-1}\sqrt{x}$, then find $\frac{dy}{dx}$. *(Delhi 2015C)* Ev

52. If $x^y+y^x=a^b$, then find $\frac{dy}{dx}$. *(AI 2017)* Ev

53. Differentiate $x^{\sin x}+(\sin x)^{\cos x}$ with respect to x. *(AI 2016)* Ap

54. If $x^my^n=(x+y)^{m+n}$, prove that $\frac{dy}{dx}=\frac{y}{x}$. *(Foreign 2014)* Ev

55. If $(x-y)\cdot e^{\frac{x}{x-y}}=a$, prove that $y\frac{dy}{dx}+x=2y$. *(Delhi 2014C)* Ev

56. If $(\tan^{-1}x)^y+y^{\cot x}=1$, then find $\frac{dy}{dx}$. *(AI 2014C)* Ev

5.6 Derivatives of Functions in Parametric Forms

VSA (1 mark)

57. If $x=e^t\sin t$, $y=e^t\cos t$, then the value of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$ is ______. *(2020C)* Ev

SA I (2 marks)

58. If $x=a\sec\theta$, $y=b\tan\theta$, then find $\frac{dy}{dx}$ at $\theta=\frac{\pi}{3}$. *(2020)* Ev

59. Find the differential of $\sin^2 x$ w.r.t. $e^{\cos x}$. *(NCERT, 2020)* Ap

SA II (3 marks)

60. Differentiate $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ w.r.t. $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$. *(2023)*

LA I (4 marks)

61. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ with respect to $\cos^{-1}x^2$. *(AI 2019)* Ev

62. If $x=a(2\theta-\sin 2\theta)$ and $y=a(1-\cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta=\frac{\pi}{3}$. *(2018)* Ev

63. If $x=a\sin 2t(1+\cos 2t)$ and $y=b\cos 2t(1-\cos 2t)$, find the values of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$ and $t=\frac{\pi}{3}$. *(Delhi 2016, AI 2016)* Ap

OR

If $x=a\sin 2t\,(1+\cos 2t)$ and $y=b\cos 2t(1-\cos 2t)$, show that at $t=\frac{\pi}{4}$, $\left(\frac{dy}{dx}\right)=\frac{b}{a}$. *(NCERT Exemplar, AI 2014)* Ev

64. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\frac{2x}{1+x^2}$, if $x\in(-1,1)$ *(Foreign 2016, Delhi 2014)* Ev

65. If $x=ae^t(\sin t+\cos t)$ and $y=ae^t(\sin t-\cos t)$, prove that $\frac{dy}{dx}=\frac{x+y}{x-y}$. *(AI 2015C)* Ap

66. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, when $x\neq 0$. *(Delhi 2014)* Ev

67. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$. *(Delhi 2014)* Ev

68. Find the value of $\frac{dy}{dx}$ at $\theta=\frac{\pi}{4}$, if $x=ae^\theta(\sin\theta-\cos\theta)$ and $y=ae^\theta(\sin\theta+\cos\theta)$. *(AI 2014)* Ap

69. If $x=\cos t(3-2\cos^2 t)$ and $y=\sin t\,(3-2\sin^2 t)$, find the value of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$. *(AI 2014)* Ev

5.7 Second Order Derivative

MCQ

70. If $x=A\cos 4t+B\sin 4t$, then $\frac{d^2x}{dt^2}$ is equal to
(a) x (b) $-x$ (c) $16x$ (d) $-16x$
(2023)

71. If $y = e^{-x}$, then $\frac{d^2y}{dx^2}$ is equal to
(a) $-y$ (b) y (c) x (d) $-x$
(Term I, 2021-22) U

72. If $x = t^2 + 1$, $y = 2at$, then $\frac{d^2y}{dx^2}$ at $t = a$ is
(a) $-\frac{1}{a}$ (b) $-\frac{1}{2a^2}$ (c) $\frac{1}{2a^2}$ (d) 0
(Term I, 2021-22) Ev

73. If $y = \sin(2\sin^{-1}x)$, then $(1 - x^2)\, y_2$ is equal to
(a) $-xy_1 + 4y$ (b) $-xy_1 - 4y$
(c) $xy_1 - 4y$ (d) $xy_1 + 4y$
(Term I, 2021-22) Ap

74. If $y = \log_e\left(\frac{x^2}{e^2}\right)$, then $\frac{d^2y}{dx^2}$ equals
(a) $-\frac{1}{x}$ (b) $-\frac{1}{x^2}$ (c) $\frac{2}{x^2}$ (d) $-\frac{2}{x^2}$
(2020) Ev

SA I (2 marks)

75. If $x = at^2$, $y = 2at$, then find $\frac{d^2y}{dx^2}$. (2020) Ev

76. If $x = a\cos\theta$; $y = b\sin\theta$, then find $\frac{d^2y}{dx^2}$. (2020) Ap

SA II (3 marks)

77. If $y = \tan x + \sec x$, then prove that
$\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$. (2023)

78. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, then show that $\frac{dy}{dx} = -\frac{x}{y}$ and hence show that
$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$. (2021C) Ap

LA I (4 marks)

79. If $x = a\sec^3\theta$, $y = a\tan^3\theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$.
(2020, Delhi 2015C) Ev

80. If $y = a\cos(\log x) + b\sin(\log x)$, show that
$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$. (2019C) Ap

81. If $y = (\sin^{-1}x)^2$, prove that
$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$. (Delhi 2019) Ap

82. If $x = \sin t$, $y = \sin pt$, prove that
$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$. (AI 2019, Foreign 2016) Ap

83. If $y = \sin(\sin x)$, prove that
$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2x = 0$. (2018) Ap

84. If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$.
(Delhi 2017) Ev

85. If $e^y(x + 1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.
(AI 2017) Ap

86. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.
(Delhi 2016, 2014) Ap

87. If $y = 2\cos(\log x) + 3\sin(\log x)$, prove that
$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ (AI 2016) Ap

88. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, show that
$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ (Delhi 2015, Foreign 2014) Ap

89. If $y = e^{m\sin^{-1}x}$, $-1 \le x \le 1$, then show that
$(1-x^2)\frac{d^2y}{dx^2} - \frac{x\,dy}{dx} - m^2y = 0$. (AI 2015) Ap

90. If $y = (x + \sqrt{1+x^2})^n$, then show that
$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y$. (Foreign 2015) Ap

91. If $y = Ae^{mx} + Be^{nx}$, show that
$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$. (AI 2015C, 2014) Ap

92. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. (Delhi 2014C) Ev

93. If $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$, evaluate
$\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$. (Delhi 2014C) Ev

CBSE Sample Questions

5.2 Continuity

MCQ

1. The value of 'k' for which the function
$$f(x) = \begin{cases} \frac{1-\cos 4x}{8x^2}, & \text{if } x \ne 0 \\ k, & \text{if } x = 0 \end{cases}$$
is continuous at $x = 0$ is
(a) 0 (b) -1 (c) 1 (d) 2
(2022-23) Ap

2. The value of $k(k < 0)$ for which the function f defined as $f(x)=\begin{cases}\frac{1-\cos kx}{x\sin x}, & x\neq 0\\ \frac{1}{2}, & x=0\end{cases}$ is continuous at $x = 0$ is

(a) ± 1 (b) -1 (c) $\pm\frac{1}{2}$ (d) $\frac{1}{2}$

(Term I, 2021-22) Ap

3. The point(s), at which the function f given by $f(x)=\begin{cases}\frac{x}{|x|}, x<0\\ -1, x\geq 0\end{cases}$ is continuous, is/are

(a) $x \in R$ (b) $x = 0$
(c) $x \in R - \{0\}$ (d) $x = -1$ and 1

(Term I, 2021-22) Ap

SA I (2 marks)

4. Find the value(s) of k so that the following function is continuous at $x = 0$.

$f(x)=\begin{cases}\frac{1-\cos kx}{x\sin x} & \text{if } x\neq 0\\ \frac{1}{2} & \text{if } x=0\end{cases}$ (2020-21) Ap

5.3 Differentiability

SA I (2 marks)

5. If $y\sqrt{1-x^2}+x\sqrt{1-y^2}=1$, then prove that $\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$. (2022-23) Ev

SA II (3 marks)

6. Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x = 1$.

(2020-21) Ap

5.4 Exponential and Logarithmic Functions

MCQ

7. If $e^x + e^y = e^{x+y}$, then $\frac{dy}{dx}$ is

(a) e^{y-x} (b) e^{y+x} (c) $-e^{y-x}$ (d) $2e^{x-y}$

(Term I, 2021-22) Ev

8. If $y = \log(\cos e^x)$, then $\frac{dy}{dx}$ is

(a) $\cos e^{x-1}$ (b) $e^{-x}\cos e^x$
(c) $e^x \sin e^x$ (d) $-e^x \tan e^x$

(Term I, 2021-22) Ev

5.5 Logarithmic Differentiation

SA I (2 marks)

9. If $y=e^{x\sin^2 x}+(\sin x)^x$, find $\frac{dy}{dx}$. (2020-21) Ap

5.6 Derivatives of Functions in Parametric Forms

MCQ

10. The derivative of $\sin^{-1}(2x\sqrt{1-x^2})$ w.r.t. $\sin^{-1}x$, $\frac{1}{\sqrt{2}}<x<1$, is

(a) 2 (b) $\frac{\pi}{2}-2$ (c) $\frac{\pi}{2}$ (d) -2

(Term I, 2021-22) Ev

5.7 Second Order Derivative

MCQ

11. If $y = 5\cos x - 3\sin x$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $-y$ (b) y (c) $25y$ (d) $9y$

(Term I, 2021-22) Ap

12. If $x = a\sec\theta$, $y = b\tan\theta$, then $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{6}$ is

(a) $\frac{-3\sqrt{3}b}{a^2}$ (b) $\frac{-2\sqrt{3}b}{a}$ (c) $\frac{-3\sqrt{3}b}{a}$ (d) $\frac{-b}{3\sqrt{3}a^2}$

(Term I, 2021-22) Ap

SA II (3 marks)

13. If $x = a\sec\theta$, $y = b\tan\theta$, find $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{6}$.

(2020-21) Ap

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. (b): Let $x = 1.5$

$\therefore$ L.H.L. $= \underset{x\to 1.5^-}{\text{Lt}} f(x) = \underset{h\to 0}{\text{Lt}}[1.5-h]=1$

and R.H.L. $= \underset{x\to 1.5^+}{\text{Lt}} f(x) = \underset{h\to 0}{\text{Lt}}[1.5+h]=1$

$\because$ L. H. L. = R. H. L.

$\therefore$ $f(x)$ is continuous at $x = 1.5$

Also, greatest integer function is discontinuous at all integral values of x.

2. (b): Since $f(x)$ is continuous at $x = 5$,

$\Rightarrow \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$

$\Rightarrow 3(5) - 8 = 2k \Rightarrow 7 = 2k \Rightarrow k=\frac{7}{2}$

3. (c): Since, greatest integer function *i.e.*, $[x]$ is continuous at all points except at integers.

$\therefore$ $f(x)$ is continuous at 1.5.

4. $\because$ $f(x)$ is continuous at $x = \pi$

$\therefore$ $f(\pi) = \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x)$ ...(i)

Here, $f(\pi) = \lambda\pi$ ...(ii)

And $\lim_{x\to\pi^+} f(x) = \lim_{h\to 0} f(\pi+h)$

$= \lim_{h\to 0} \cos(\pi+h) = -1$ ...(iii)

From (i), (ii) and (iii), we get

$\lambda\pi = -1 \Rightarrow \lambda = -\frac{1}{\pi}$

Concept Applied

- A real function f is said to be continuous if it is continuous at every point in the domain of f.

5. We have, $f(x) = \begin{cases} \frac{kx}{|x|}, & x<0 \\ 3, & x\ge 0 \end{cases}$

L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{kx}{-x} = -k$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3 = 3$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ L.H.L. = R.H.L. = $f(0)$

$\Rightarrow$ $-k = 3 \Rightarrow k = -3$

6. Given, $f(x)$ is continuous at $x = 3$.

So, $\lim_{x\to 3} f(x) = f(3) \Rightarrow \lim_{x\to 3} \frac{(x+3)^2-36}{x-3} = k$

$\Rightarrow \lim_{x\to 3} \frac{(x+3)^2-6^2}{x-3} = k \Rightarrow \lim_{x\to 3} \frac{(x+3+6)(x+3-6)}{x-3} = k$

$\Rightarrow$ $3+3+6 = k \Rightarrow k = 12$

7. We have, $f(x) = \begin{cases} \frac{\sin^2 \lambda x}{x^2}, & x\ne 0 \\ 1, & x=0 \end{cases}$ is continuous at $x = 0$

$\therefore$ L.H.L. = R.H.L.

$\Rightarrow \underset{x\to 0}{\text{Lt}} \frac{\sin^2\lambda x}{x^2} = 1 \Rightarrow \underset{x\to 0}{\text{Lt}} \frac{\sin^2\lambda x}{\lambda^2 x^2}\cdot\lambda^2 = 1$

$\Rightarrow \lambda^2 \underset{\lambda x\to 0}{\text{Lt}} \left(\frac{\sin\lambda x}{\lambda x}\right)^2 = 1 \Rightarrow \lambda^2\cdot 1 = 1 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

8. We have, $f(x) = \begin{cases} ax+1 & \text{if } x\le 3 \\ bx+3 & \text{if } x>3 \end{cases}$

Since, $f(x)$ is continuos at $x = 3$

$\therefore$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \Rightarrow \lim_{x\to 3} ax+1 = \lim_{x\to 3} bx+3$

$\Rightarrow$ $3a+1 = 3b+3 \Rightarrow 3a-3b = 2 \Rightarrow a-b = 2/3$

9. $\because$ $f(x)$ is continuous at $\pi/2$.

$\therefore$ $\lim_{x\to\pi/2^-} f(x) = \lim_{x\to\pi/2^+} f(x) = f(\pi/2)$ ...(1)

Now, $\lim_{x\to\pi/2^-} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$

$= \lim_{h\to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)} = \lim_{h\to 0} \frac{1-\cos^3 h}{3\sin^2 h}$

$= \lim_{h\to 0} \frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$

$= \lim_{h\to 0} \frac{(1+\cos^2 h+\cos h)}{3(1+\cos h)} = \frac{1+1+1}{3(1+1)} = \frac{1}{2}$

and $\lim_{x\to\pi/2^+} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$

$= \lim_{h\to 0} \frac{q\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2} = \lim_{h\to 0} \frac{q(1-\cos h)}{4h^2}$

$= \frac{q}{4}\times \lim_{h\to 0} \frac{2\sin^2\frac{h}{2}}{\frac{h^2}{4}\times 4} = \frac{q}{4}\times\frac{2}{4} = \frac{q}{8}$ and $f(\pi/2) = p$

$\therefore$ $\frac{1}{2} = \frac{q}{8} = p$ [From (1)]

$\Rightarrow$ $p = \frac{1}{2}$ and $q = 4$

Commonly Made Mistake

- Remember the difference between left hand limit and right hand limit.

10. $\because$ $f(x)$ is continuous at $x = 0$.

$\therefore$ $f(0) = k$

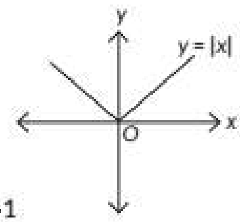
and $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos 4x}{8x^2}$

$= \lim_{x\to 0} \frac{2\cdot\sin^2 2x}{8x^2} = \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1$

$\because$ f is continuous at $x = 0$

$\therefore$ $f(0) = \lim_{x\to 0} f(x) \Rightarrow k = 1$

11. (c): $f(x) = |x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$



The function $f(x)$ is continuous everywhere but not differentiable at $x = 0$ as at $x = 0$

$Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{-x-0}{x} = -1$

$Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x} = 1$

$\therefore$ $Lf'(0) \ne Rf'(0)$, so $f(x)$ is not differentiable at $x = 0$.

12. (c): Let $y = x^{2x}$

Taking log on both sides, we get

$\log y = 2x \log x$

Differentiating both sides w.r.t. x, we get

$\frac{1}{y}\frac{dy}{dx} = 2\left\{x\cdot\frac{1}{x} + \log x\cdot 1\right\}$

$\Rightarrow$ $\frac{dy}{dx} = 2y\{1+\log x\} = 2x^{2x}(1+\log x)$

13. (a): Given, $y^2(2-x) = x^3$

$\Rightarrow$ $y^2 = \frac{x^3}{2-x} \Rightarrow 2y\cdot\frac{dy}{dx} = \frac{(2-x)\times 3x^2 - x^3(-1)}{(2-x)^2}$

$\Rightarrow \frac{dy}{dx}=\frac{6x^2-2x^3}{2y(2-x)^2} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)}=\frac{6-2}{2\times1}=2$

14. (a): At $x = 1$ $\lim_{x\to1^-} f(x)=\lim_{x\to1} x^2=1$

And $\lim_{x\to1^+} f(x)=\lim_{x\to1} 2-x=1$

Also, $f(1) = 2 - 1 = 1$ $\because \lim_{x\to1^-} f(x)=\lim_{x\to1^+} f(x)=f(x)$

$\therefore$ $f(x)$ is continuous at $x = 1$

Now, L.H.D. $=\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1}\frac{x^2-1}{x-1}=\lim_{x\to1}(x+1)=2$

R.H.D. $=\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1}\frac{(2-x)-1}{x-1}=-1$

$\because$ L.H.D. $\neq$ R.H.D. $\therefore$ $f(x)$ is not differentiable at $x = 1$.

Commonly Made Mistake

- Every continuous function is not differentiable.

15. (c): Given, $\sec^{-1}\left(\frac{1+x}{1-y}\right)=a \Rightarrow \sec a=\frac{1+x}{1-y}$

On differentiating, we get

$\frac{(1-y)+(1+x)\frac{dy}{dx}}{(1-y)^2}=0 \Rightarrow (1+x)\frac{dy}{dx}=y-1 \Rightarrow \frac{dy}{dx}=\frac{y-1}{1+x}$

16. We have, $y = \tan^{-1} x + \cot^{-1} x$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x^2+1}-\frac{1}{x^2+1} \Rightarrow \frac{dy}{dx}=0$

17. We have, $\cos(xy) = k$

$\Rightarrow -\sin(xy)\left(y+x\frac{dy}{dx}\right)=0 \Rightarrow y+x\frac{dy}{dx}=0$ $[\because xy\neq n\pi]$

$\Rightarrow \frac{dy}{dx}=-\frac{y}{x}$

18. Let $y=\sin^2(\sqrt{x})$

$\therefore \frac{dy}{dx}=2\sin(\sqrt{x})\cdot\cos(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}}\sin(2\sqrt{x})$

Key Points

- $\sin 2x = 2\sin x\cos x$

19. To check the differentiability of $f(x) = x|x|$ at $x = 0$.

Consider, $\lim_{h\to0}\frac{f(0+h)-f(0)}{h}$

$=\lim_{h\to0}\frac{f(h)-f(0)}{h}=\lim_{h\to0}\frac{h|h|-0}{h}=\lim_{h\to0}|h|=0$

Hence, $f'(0)$ exists, so $f(x) = x|x|$ is differentiable at $x = 0$.

Answer Tips

- $f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

20. We have, $y = f(x^2) \Rightarrow \frac{dy}{dx}=f'(x^2)\cdot 2x$

$=e^x\cdot 2x$ $[\because f'(x)=e^{\sqrt{x}}]$

$= 2xe^x$

21. Given, $f(x) = x + 1$

Now, $(fof)(x) = f(f(x)) = f(x + 1) = (x + 1) + 1 = x + 2$

$\therefore \frac{d}{dx}(fof)(x)=\frac{d}{dx}(x+2)=1$

22. We have, $(x^2 + y^2)^2 = xy$

$\Rightarrow x^4 + y^4 + 2x^2y^2 = xy$

On differentiating both sides w.r.t. x, we get

$4x^3+4y^3\frac{dy}{dx}+4xy^2+4x^2y\frac{dy}{dx}=x\frac{dy}{dx}+y$

$\Rightarrow \frac{dy}{dx}(4y^3+4x^2y-x)=y-4x^3-4xy^2$

$\Rightarrow \frac{dy}{dx}=\frac{y-4x^3-4xy^2}{4y^3+4x^2y-x}$

23. We have, $f(x)=\begin{cases}x^2, \text{ if } x\geq1\\ x, \text{ if } x<1\end{cases}$

R.H.D. $=\operatorname{Lt}_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\operatorname{Lt}_{x\to1}\frac{x^2-1}{x-1}=\operatorname{Lt}_{x\to1}\frac{(x-1)(x+1)}{x-1}$

$=\operatorname{Lt}_{x\to1}(x+1)=2$

L.H.D. $=\operatorname{Lt}_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\operatorname{Lt}_{x\to1}\frac{x-1}{x-1}=1$

$\because$ L.H.D. $\neq$ R.H.D.

$\therefore$ $f(x)$ is not differentiable at $x = 1$

24. We have, $f(x) = |x - 3|$

$f(x)=\begin{cases}x-3, & x\geq3\\ -x+3, & x<3\end{cases}$

At $x = 3$

$f'(3^+)=\lim_{h\to0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to0}\frac{3+h-3-(3-3)}{h}=\lim_{h\to0}1=1$

$f'(3^-)=\lim_{h\to0}\frac{f(3-h)-f(3)}{-h}=\lim_{h\to0}\frac{-3+h+3-(0)}{-h}=\lim_{h\to0}-1=-1$

$\because$ $f'(3^+)\neq f'(3^-)$ $\therefore$ $f(x)$ is not differentiable at $x = 3$.

25. We have, $y=\sqrt{a+\sqrt{a+x}}$

$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{1}{\sqrt{a+\sqrt{a+x}}}\times\frac{1}{2}\frac{1}{\sqrt{a+x}}=\frac{1}{4}\cdot\frac{1}{\sqrt{a\sqrt{(a+x)}+a+x}}$

26. Let, $y=\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left[\tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)\right]$

$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right) \Rightarrow \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\left(\cot\frac{x}{2}\right)$

$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right] \Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right)=-\frac{1}{2}$

27. We have, $\sin^2 y + \cos xy = K$

Differentiating w.r.t. x on both sides, we get

$2\sin y\cos y\frac{dy}{dx}+(-\sin xy)\left(x\frac{dy}{dx}+y\right)=0$

$(\because$ Product rule : $(uv)'=u'v+uv')$

$\Rightarrow \frac{dy}{dx}=\frac{y\sin xy}{\sin 2y-x\sin xy}$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{\left(1,\frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{\pi}{2}-\sin\frac{\pi}{4}} = \frac{\pi}{4(\sqrt{2}-1)}$$

28. Given $f(x)=\begin{cases} |x-3|, & x\ge 1 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1 \end{cases}$

$$f(x)=\begin{cases} -(x-3), & 1\le x<3 \\ x-3, & x\ge 3 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1 \end{cases}$$

(i) R.H.D. at $x = 1$

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{-(1+h-3)-[-(1-3)]}{h}$$

$$=\lim_{h\to 0}\frac{(-h+2-2)}{h}=\lim_{h\to 0}\frac{-h}{h}=-1$$

(ii) L.H.D. at $x = 1$

$$f'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$=\lim_{h\to 0}\left[\frac{\left[\frac{(1-h)^2}{4}-\frac{3}{2}(1-h)+\frac{13}{4}\right]-\left[\frac{1}{4}-\frac{3}{2}+\frac{13}{4}\right]}{-h}\right]$$

$$=\lim_{h\to 0}\left[\frac{\frac{1+h^2-2h}{4}-\frac{3}{2}+\frac{3h}{2}+\frac{13}{4}-\frac{1}{4}+\frac{3}{2}-\frac{13}{4}}{-h}\right]$$

$$=\lim_{h\to 0}\left[\frac{\frac{h^2}{4}-\frac{h}{2}+\frac{3h}{2}}{-h}\right]=\lim_{h\to 0}\left[\frac{h^2}{-4h}+\frac{h}{-h}\right]=0-1=-1$$

(iii) (a) We know, that function $f(x)$ is differentiable at $x = 1$ if L.H.D. = R.H.D. = $f'(1)$

$\Rightarrow\ -1 = -1 = -1$

Hence, the given function $f(x)$ is differentiable at $x = 1$.

OR

(b) Differentiate $f(x)$ w.r.t. x, we get

$$f'(x)=\begin{cases} -1, & 1\le x<3 \\ 1, & x\ge 3 \\ \frac{x}{2}-\frac{3}{2}, & x<1 \end{cases}$$

$$f'(2)=-1 \text{ and } f'(-1)=\frac{-1}{2}-\frac{3}{2}=\frac{-4}{2}=-2$$

29. Given that $f(x)$ is differentiable at $x = 1$. Therefore, $f(x)$ is continuous at $x = 1$

$$\Rightarrow \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$$

$$\Rightarrow \lim_{x\to 1}(x^2+3x+a)=\lim_{x\to 1}(bx+2)$$

$$\Rightarrow 1+3+a=b+2 \Rightarrow a-b+2=0 \quad \text{...(1)}$$

Again, $f(x)$ is differentiable at $x = 1$. So,
(L.H.D. at $x = 1$) = (R.H.D. at $x = 1$)

$$\Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x^2+3x+a)-(4+a)}{x-1}=\lim_{x\to 1}\frac{(bx+2)-(4+a)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{x^2+3x-4}{x-1}=\lim_{x\to 1}\frac{bx-2-a}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x+4)(x-1)}{x-1}=\lim_{x\to 1}\frac{bx-b}{x-1} \quad \text{[From (1)]}$$

$$\Rightarrow \lim_{x\to 1}(x+4)=\lim_{x\to 1}\frac{b(x-1)}{x-1} \Rightarrow 5=b$$

Putting $b = 5$ in (1), we get $a = 3$. Hence, $a = 3$ and $b = 5$

Key Points

- If a function f is differentiable at a point c, then it is also continuous at that point.

30. We have,

$$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2\le 1$$

Putting $x^2 = \cos\theta \Rightarrow \theta = \cos^{-1}(x^2)$ we get

$$y=\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)=\tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)=\frac{\pi}{4}+\frac{\theta}{2} \Rightarrow y=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

Differentiating w.r.t. x on both sides, we get

$$\frac{dy}{dx}=-\frac{1\times 2x}{2\sqrt{1-x^4}}=\frac{-x}{\sqrt{1-x^4}}$$

31. Here $f(x)=\sqrt{x^2+1}=(x^2+1)^{\frac{1}{2}}$

$$\Rightarrow f'(x)=\frac{1}{2}\cdot(x^2+1)^{-\frac{1}{2}}\cdot 2x=\frac{x}{\sqrt{x^2+1}}, \quad \text{...(1)}$$

and $g(x)=\dfrac{x+1}{x^2+1}$

$$\Rightarrow g'(x)=\frac{(x^2+1)\cdot 1-(x+1)\cdot 2x}{(x^2+1)^2}=\frac{-x^2-2x+1}{(x^2+1)^2} \quad \text{...(2)}$$

and $h(x) = 2x - 3 \Rightarrow h'(x) = 2$ ...(3)

$$\therefore f'[h'(g'(x))]=f'\left[h'\left(\frac{-x^2-2x+1}{(x^2+1)^2}\right)\right] \quad \text{[Using (2)]}$$

$$=f'(2) \quad \text{[Using (3)]}$$

$$=\frac{2}{\sqrt{2^2+1}}=\frac{2}{\sqrt{5}} \quad \text{[Using (1)]}$$

Concept Applied

- Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{du}\times\frac{du}{dx}$

32. The given function is $f(x) = |x - 1| + |x + 1|$

$$=\begin{cases} -(x-1)-(x+1), & x<-1 \\ -(x-1)+x+1, & -1\le x\le 1 \\ x-1+x+1, & x>1 \end{cases}=\begin{cases} -2x, & x<-1 \\ 2, & -1\le x\le 1 \\ 2x, & x>1 \end{cases}$$

At $x = 1$,

$$f'(1^-)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{2-2}{-h}=0$$

$$f'(1^+)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{2(1+h)-2}{h}=\lim_{h\to 0}\frac{2h}{h}=2$$

$$\therefore\ f'(1^-)\ne f'(1^+)$$

$\Rightarrow$ f is not differentiable at $x = 1$.

At $x = -1$,

$$f'(-1^-)=\lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h}$$

$$=\lim_{h\to 0}\frac{-2(-1-h)-(2)}{-h}=\lim_{h\to 0}\frac{2h}{-h}=-2$$

$$f'(-1^+)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0}\frac{2-2}{h}=0$$

$\therefore$ $f'(-1^-)\neq f'(-1^+)$

$\Rightarrow$ f is not differentiable at $x = -1$.

33. At $x = 1$

$$f'(1^-)=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{x-1}{x-1}=1$$

$$f'(1^+)=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{2-x-1}{x-1}=\lim_{x\to 1}\frac{1-x}{x-1}=-1$$

Since, $f'(1^-)\neq f'(1^+)$

$\therefore$ f(x) is not differentiable at $x = 1$.

At $x = 2$

$$f'(2^-)=\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{2-x-0}{x-2}=-1$$

$$f'(2^+)=\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$

$$=\lim_{x\to 2}\frac{-2+3x-x^2-0}{x-2}=\lim_{x\to 2}\frac{(1-x)(x-2)}{x-2}=-1$$

Since, $f'(2^-) = f'(2^+)$

$\therefore$ f(x) is differentiable at $x = 2$.

34. Here $f(x)=\begin{cases}\lambda(x^2+2), & \text{if } x\le 0\\ 4x+6, & \text{if } x>0\end{cases}$

At $x = 0$, $f(0) = \lambda(0^2 + 2) = 2\lambda$

$$\text{L.H.L}=\lim_{x\to 0^-}f(x)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}\lambda[(0-h)^2+2]=2\lambda$$

$$\text{R.H.L}=\lim_{x\to 0^+}f(x)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}[4(0+h)+6]=6$$

$\therefore$ For f to be continuous at $x = 0$

$2\lambda = 6 \Rightarrow \lambda = 3$.

Hence the function becomes

$$f(x)=\begin{cases}3(x^2+2), & \text{if } x\le 0\\ 4x+6 & \text{if } x>0\end{cases}$$

$$f'(0^-)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h}=\lim_{h\to 0}\frac{3(h^2+2)-6}{-h}=\lim_{h\to 0}(-3h)=0$$

$$\text{and } f'(0^+)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h}=\lim_{h\to 0}\frac{4h+6-6}{h}=4$$

$\Rightarrow$ $f'(0^-)\neq f'(0^+)$ $\therefore$ f is not differentiable at $x = 0$.

35. We have $\cos y = x\cos(a + y)$

$$\Rightarrow x=\frac{\cos y}{\cos(a+y)}$$

Differentiating w.r.t. y on both sides, we get

$$\frac{dx}{dy}=\frac{\cos(a+y)\left(\frac{d}{dy}\cos y\right)-\cos y\left(\frac{d}{dy}\cos(a+y)\right)}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)+\cos y\sin(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy}=\frac{\cos y\sin(a+y)-\cos(a+y)\sin y}{\cos^2(a+y)}$$

$$=\frac{\sin[(a+y)-y]}{\cos^2(a+y)}=\frac{\sin a}{\cos^2(a+y)} \quad \therefore \quad \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

Concept Applied

- Quotient rule : $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$

36. We have, $y=\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)$

$$\Rightarrow y=\sin^{-1}\left(x\sqrt{1-(\sqrt{x})^2}-\sqrt{x}\sqrt{1-x^2}\right)$$

$$\Rightarrow y=\sin^{-1}x-\sin^{-1}\sqrt{x}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{d}{dx}(\sqrt{x}) \quad \left(\because \frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}\right)$$

$$=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2}\cdot\frac{1}{\sqrt{x-x^2}}$$

37. (c) : $y = \log(\sin e^x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\sin e^x}\cdot\frac{d}{dx}(\sin e^x)=\frac{1}{\sin e^x}\cos e^x\cdot\frac{d}{dx}e^x=\frac{1}{\sin e^x}\cos e^x\cdot e^x$$

$$=e^x\cot e^x$$

38. (a): Given, $y = \tan^{-1}(e^{2x})$

$$\therefore \frac{dy}{dx}=\frac{1}{1+e^{4x}}\times 2e^{2x}=\frac{2e^{2x}}{1+e^{4x}}$$

Answer Tips

- $\frac{d}{dx}(e^x)=e^x$

39. Given $y = \log(\cos e^x)$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\cos e^x}(-\sin e^x\cdot e^x)=-e^x\tan e^x$$

40. We have, $y = e^{x^2\cos x} + (\cos x)^x$

$$= e^{x^2\cos x} + e^{x(\ln\cos x)}$$

$$\therefore \frac{dy}{dx}=e^{x^2\cos x}\frac{d}{dx}(x^2\cos x)+e^{x\ln\cos x}\frac{d}{dx}(x\ln\cos x)$$

$$=e^{x^2\cos x}(2x\cos x-x^2\sin x)+e^{x\ln\cos x}\left(\ln\cos x-\frac{x}{\cos x}\sin x\right)$$

$$=e^{x^2\cos x}(2x\cos x-x^2\sin x)+(\cos x)^x(\ln\cos x-x\tan x)$$

41. Given, $\log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)$

On differentiating w.r.t. x on both sides, we get

$$\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right)=2\times\frac{1}{1+\frac{y^2}{x^2}}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{2x}{x^2+y^2}+\frac{2y}{x^2+y^2}\frac{dy}{dx}=\frac{2x^2}{x^2+y^2}\left(\frac{1}{x}\frac{dy}{dx}+y\left(\frac{-1}{x^2}\right)\right)$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{2y}{x^2+y^2}-\frac{2x}{x^2+y^2}\right]=\frac{2x^2}{x^2+y^2}\left[\frac{-y}{x^2}-\frac{1}{x}\right]$$

$$\Rightarrow \frac{2(y-x)}{x^2+y^2}\frac{dy}{dx}=\frac{-2x^2}{x^2+y^2}\left(\frac{y+x}{x^2}\right) \Rightarrow \frac{dy}{dx}=\frac{x+y}{x-y}$$

Key Points

- To differentiate an implicit function, consider y as a function of x then apply derivative rules.

42. Here $y=\frac{x\cos^{-1}x}{\sqrt{1-x^2}}-\log\sqrt{1-x^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{\sqrt{1-x^2}\cdot\frac{d}{dx}(x\cos^{-1}x)-x\cos^{-1}x\cdot\frac{d}{dx}\sqrt{1-x^2}}{1-x^2}-\frac{1}{\sqrt{1-x^2}}\cdot\frac{d}{dx}(\sqrt{1-x^2})$$

$$=\frac{\sqrt{1-x^2}\cdot\left(1\cdot\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}\right)-x\cos^{-1}x\left(\frac{-x}{\sqrt{1-x^2}}\right)}{1-x^2}-\frac{1}{\sqrt{1-x^2}}\left(\frac{-x}{\sqrt{1-x^2}}\right)$$

$$=\frac{\sqrt{1-x^2}\cos^{-1}x-x+\frac{x^2\cos^{-1}x}{\sqrt{1-x^2}}}{1-x^2}+\frac{x}{1-x^2}$$

$$=\frac{(1-x^2)\cos^{-1}x+x^2\cos^{-1}x}{(1-x^2)\sqrt{1-x^2}}=\frac{\cos^{-1}x}{(1-x^2)^{3/2}}$$

Concept Applied

- Product rule of derivative : $(uv)'=u'v+uv'$

43. Given $e^x+e^y=e^{x+y} \Rightarrow 1+e^{y-x}=e^y$ ...(1)

Differentiating (1) w.r.t. x, we get

$$e^{y-x}\cdot\frac{d}{dx}(y-x)=e^y\frac{dy}{dx}$$

$$\Rightarrow e^{y-x}\left(\frac{dy}{dx}-1\right)=e^y\frac{dy}{dx} \Rightarrow \frac{dy}{dx}(e^{y-x}-e^y)=e^{y-x}$$

$$\Rightarrow \frac{dy}{dx}(-1)=e^{y-x} \Rightarrow \frac{dy}{dx}+e^{y-x}=0 \quad \text{[Using (1)]}$$

44. Here, $y=\tan^{-1}\left(\frac{a}{x}\right)+\log\sqrt{\frac{x-a}{x+a}}$

$$=\tan^{-1}\left(\frac{a}{x}\right)+\frac{1}{2}\log\left(\frac{x-a}{x+a}\right)$$

$$=\tan^{-1}\left(\frac{a}{x}\right)+\frac{1}{2}[\log(x-a)-\log(x+a)]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{1+\frac{a^2}{x^2}}\cdot\frac{d}{dx}\left(\frac{a}{x}\right)+\frac{1}{2}\left[\frac{1}{x-a}-\frac{1}{x+a}\right] \quad \left(\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right)$$

$$=\frac{x^2}{x^2+a^2}\cdot a\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{2}\left[\frac{(x+a)-(x-a)}{x^2-a^2}\right]$$

$$=\frac{-a}{x^2+a^2}+\frac{a}{x^2-a^2}=\frac{-a(x^2-a^2)+a(x^2+a^2)}{x^4-a^4}=\frac{2a^3}{x^4-a^4}$$

Concept Applied

- $\log\left(\frac{a}{b}\right)=\log a-\log b$

45. We have, $e^{y-x}=y^x$

Taking log on both sides

$(y-x)\log e=x\log y \Rightarrow y-x=x\log y$ ...(i)

On differentiating, we get

$$\frac{dy}{dx}-1=x\cdot\frac{1}{y}\frac{dy}{dx}+\log y \Rightarrow \frac{dy}{dx}-\frac{x}{y}\frac{dy}{dx}=1+\log y$$

$$\Rightarrow \frac{dy}{dx}\left(1-\frac{x}{y}\right)=1+\log y \Rightarrow \frac{dy}{dx}=\frac{y(1+\log y)}{y-x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y(1+\log y)}{x\log y} \quad \text{[From (i)]}$$

46. We have $y=x^3(\cos x)^x+\sin^{-1}\sqrt{x}$ ...(i)

Let $z=(\cos x)^x=e^{x\log\cos x}$

$$\therefore \frac{dz}{dx}=e^{x\log\cos x}\left[\frac{x(-\sin x)}{\cos x}+\log\cos x\right]$$

$=(\cos x)^x\times[-x\tan x+\log\cos x]$ ...(ii)

Now, differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=3x^2(\cos x)^x+x^3(\cos x)^x[-x\tan x+\log\cos x]+\frac{1}{\sqrt{1-x}}\left(\frac{1}{2\sqrt{x}}\right) \quad \text{[Using (ii)]}$$

$$=x^2(\cos x)^x[3-x^2\tan x+x\log\cos x]+\frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{1-x}}\right)$$

47. Let $y=(\log x)^x+x^{\log x}$ $\therefore$ $y=e^{x\log(\log x)}+e^{(\log x)^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=(\log x)^x\frac{d}{dx}\{x\log(\log x)\}+x^{\log x}\frac{d}{dx}\{(\log x)^2\}$$

$$=(\log x)^x\left\{x\left(\frac{1}{\log x}\right)\frac{1}{x}+\log(\log x)\right\}+x^{\log x}\left(2(\log x)\frac{1}{x}\right)$$

$$=(\log x)^x\left\{\frac{1}{\log x}+\log(\log x)\right\}+2\left(\frac{\log x}{x}\right)x^{\log x}$$

48. Given, $x^y\cdot y^x=x^x$

Taking log on both sides, we get

$y\log x+x\log y=x\log x$

$$\Rightarrow y\cdot\frac{1}{x}+\log x\frac{dy}{dx}+\frac{x}{y}\frac{dy}{dx}+\log y=x\cdot\frac{1}{x}+\log x$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{x}{y}+\log x\right)=1+\log x-\frac{y}{x}-\log y=1-\frac{y}{x}+\log\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-\frac{y}{x}+\log\frac{x}{y}}{\frac{x}{y}+\log x}$$

49. We have, $x^y-y^x=a^b$

Taking log on both sides, we get

$y\log x-x\log y=b\log a$ ...(i)

Now, differentiating (i) w.r.t. x on both sides, we get

$$\frac{y}{x}+(\log x)\frac{dy}{dx}-\log y-\frac{x\cdot1}{y}\frac{dy}{dx}=0$$

$$\Rightarrow \left(\frac{y}{x}-\log y\right)=\frac{dy}{dx}\left(\frac{x}{y}-\log x\right)$$

$$\Rightarrow \frac{1}{x}(y-x\log y)=\frac{dy}{dx}\left(\frac{x-y\log x}{y}\right) \Rightarrow \frac{dy}{dx}=\frac{y(y-x\log y)}{x(x-y\log x)}$$

50. Given $y = (x)^{\cos x} + (\cos x)^{\sin x}$

Let $u = (x)^{\cos x}$, $v = (\cos x)^{\sin x}$ $\therefore$ $y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(i)$$

Now, $u = x^{\cos x}$

$\Rightarrow$ $\log u = \cos x \log x$

Differentiating with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = \cos x \cdot \frac{1}{x} + \log x(-\sin x)$$

$$\Rightarrow \frac{du}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \log x\right] \quad ...(ii)$$

Now, $v = (\cos x)^{\sin x}$

$\Rightarrow$ $\log v = \sin x \log \cos x$

Differentiating with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{\cos x}(-\sin x) + \log \cos x \cdot \cos x$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x}\left[\frac{-\sin^2 x + \cos^2 x \log \cos x}{\cos x}\right] \quad ...(iii)$$

Putting value of (ii) and (iii) into (i), we get

$$\frac{dy}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \log x\right] + (\cos x)^{\sin x}\left[\frac{-\sin^2 x + \cos^2 x \log \cos x}{\cos x}\right]$$

51. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$\Rightarrow$ $y = u + v$ [where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$]

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(i)$$

Now, $u = (\sin x)^x$

Taking logarithm on both sides, we get $\log u = x \log \sin x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}(\cos x) + \log \sin x$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\,(x \cot x + \log \sin x) \quad ...(ii)$$

and $v = \sin^{-1}\sqrt{x}$

$$\Rightarrow \frac{dv}{dx} = \left(\frac{1}{\sqrt{1-x}}\right)\frac{1}{2\sqrt{x}} \quad ...(iii)$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\sin x)^x\,(x \cot x + \log \sin x) + \frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{1-x}}\right)$$

Answer Tips

➲ $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

52. We have, $x^y + y^x = a^b$

Taking log on both sides, we get

$y \log x + x \log y = b \log a$ ...(i)

Now, differentiating (i) w.r.t. x on both sides, we get

$$\frac{y}{x} + \log x\left(\frac{dy}{dx}\right) + \log y + x\cdot\frac{1}{y}\frac{dy}{dx} = 0 \qquad \left(\because \frac{d}{dx}(\log x) = \frac{1}{x}\right)$$

$$\Rightarrow \frac{y}{x} + \log y = -\left(\frac{x}{y} + \log x\right)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-y}{x}\left(\frac{y + x\log y}{x + y\log x}\right)$$

53. We have, $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $u = x^{\sin x}$, $v = (\sin x)^{\cos x}$

$\therefore$ $y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(i)$$

Now, $u = x^{\sin x}$ $\Rightarrow$ $\log u = \sin x \cdot (\log x)$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = \cos x \cdot (\log x) + \frac{1}{x}\cdot \sin x$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x}\left[\cos x \cdot (\log x) + \frac{1}{x}\sin x\right] \quad ...(ii)$$

Also, $v = (\sin x)^{\cos x} \Rightarrow \log v = \cos x\,(\log \sin x)$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = -\sin x(\log \sin x) + \cos x \cdot \frac{1}{\sin x}\cdot \cos x$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x - \sin^2 x(\log \sin x)}{\sin x}\right] \quad ...(iii)$$

From (i), (ii) & (iii), we get

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{x\cos x \cdot (\log x) + \sin x}{x}\right] + (\sin x)^{\cos x}\left[\frac{\cos^2 x - \sin^2 x(\log \sin x)}{\sin x}\right]$$

Key Points

For function $y(f(x))^{g(x)}$, we must take 'log' on both sides of the function to remove $g(x)$ as the power of $f(x)$.

54. Given $x^m y^n = (x + y)^{m+n}$

Taking log on both the sides, we get

$\log x^m + \log y^n = (m + n)\log (x + y)$

$\Rightarrow$ $m \log x + n \log y = (m + n) \log (x + y)$

Differentiating w.r.t. x, we get

$$m\cdot\frac{1}{x} + n\cdot\frac{1}{y}\frac{dy}{dx} = (m+n)\cdot\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx + ny - my - ny}{y(x+y)}\right) = \frac{mx + nx - mx - my}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx - my}{y(x+y)}\right) = \frac{nx - my}{x(x+y)} \quad \therefore \quad \frac{dy}{dx} = \frac{y}{x}$$

55. Here, $(x-y)\cdot e^{\frac{x}{x-y}} = a$

Taking log on both sides, we get

$$\log\left\{(x-y)\cdot e^{\frac{x}{x-y}}\right\} = \log a \Rightarrow \log(x-y) + \frac{x}{x-y} = \log a$$

Differentiating w.r.t. x, we get

$$\frac{1}{x-y}\cdot\left(1 - \frac{dy}{dx}\right) + \frac{(x-y)\cdot 1 - x\left(1 - \frac{dy}{dx}\right)}{(x-y)^2} = 0$$

$$\Rightarrow (x-y)\left(1 - \frac{dy}{dx}\right) + x - y - x\left(1 - \frac{dy}{dx}\right) = 0$$

$$\Rightarrow -y\left(1 - \frac{dy}{dx}\right) + x - y = 0 \Rightarrow y\frac{dy}{dx} + x = 2y$$

56. Here, $(\tan^{-1}x)^y + y^{\cot x} = 1$

$\Rightarrow$ $u + v = 1$ where $u = (\tan^{-1}x)^y$ and $v = y^{\cot x}$

Differentiating w.r.t. x, we get

$$\frac{du}{dx}+\frac{dv}{dx}=0 \quad ...(1)$$

Now, $u = (\tan^{-1}x)^y$

$\Rightarrow$ $\log u = y \log(\tan^{-1}x)$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{dy}{dx}\log(\tan^{-1}x)+y\cdot\frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}$$

$$\Rightarrow \frac{du}{dx}=(\tan^{-1}x)^y\times\left[\frac{dy}{dx}\log(\tan^{-1}x)+\frac{y}{(1+x^2)\tan^{-1}x}\right] \quad ...(2)$$

And $v = y^{\cot x}$

$\Rightarrow$ $\log v = \cot x \cdot \log y$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx}=\cot x\cdot\frac{1}{y}\cdot\frac{dy}{dx}-\operatorname{cosec}^2 x\cdot\log y$$

$$\Rightarrow \frac{dv}{dx}=y^{\cot x}\left[\frac{\cot x}{y}\cdot\frac{dy}{dx}-\operatorname{cosec}^2 x\cdot\log y\right] \quad ...(3)$$

From (1), (2) and (3), we get

$$(\tan^{-1}x)^y\left[\frac{dy}{dx}\log(\tan^{-1}x)+\frac{y}{(1+x^2)\tan^{-1}x}\right]+y^{\cot x}\left[\frac{\cot x}{y}\cdot\frac{dy}{dx}-\operatorname{cosec}^2x\cdot\log y\right]=0$$

$$\Rightarrow \frac{dy}{dx}[(\tan^{-1}x)^y\cdot\log(\tan^{-1}x)+y^{\cot x-1}\cdot\cot x]$$

$$=y^{\cot x}\cdot\operatorname{cosec}^2x\cdot\log y-(\tan^{-1}x)^{y-1}\cdot\frac{y}{(1+x^2)}$$

$$\therefore \frac{dy}{dx}=\frac{y^{\cot x}\cdot\operatorname{cosec}^2x\cdot\log y-(\tan^{-1}x)^{y-1}\cdot\frac{y}{(1+x^2)}}{(\tan^{-1}x)^y\log(\tan^{-1}x)+y^{\cot x-1}\cot x}$$

57. We have, $x = e^t \sin t$

$$\Rightarrow \frac{dx}{dt}=e^t\sin t+e^t\cos t$$

and $y = e^t\cos t \Rightarrow \frac{dy}{dt}=e^t\cos t-e^t\sin t$

$$\frac{dy}{dx}=\frac{e^t\cos t-e^t\sin t}{e^t(\cos t+\sin t)}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}=\frac{\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)}=\frac{\frac{1}{\sqrt2}-\frac{1}{\sqrt2}}{\frac{1}{\sqrt2}+\frac{1}{\sqrt2}}=0$$

58. We have, $x = a\sec\theta$, $y = b\tan\theta$

$$\therefore \frac{dx}{d\theta}=a\sec\theta\tan\theta \text{ and } \frac{dy}{d\theta}=b\sec^2\theta$$

$$\text{Now, } \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b}{a}\cdot\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}=\frac{b}{a}\cdot\frac{1}{\sin\theta}$$

$$\therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}=\frac{b}{a}\cdot\frac{1}{\sin\frac{\pi}{3}}=\frac{b}{a}\cdot\frac{1}{\frac{\sqrt3}{2}}=\frac{2b}{a\sqrt3}$$

59. Let $u = \sin^2 x$, $v = e^{\cos x}$

$$\therefore \frac{du}{dx}=2\sin x\cos x \text{ and } \frac{dv}{dx}=e^{\cos x}(-\sin x)$$

$$\text{Now, } \frac{du}{dv}=\frac{du}{dx}\cdot\frac{dx}{dv}=\frac{2\sin x\cos x}{e^{\cos x}(-\sin x)}=\frac{-2\cos x}{e^{\cos x}}$$

60. Let $u=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ ...(i)

and $v=\sin^{-1}(2x\sqrt{1-x^2})$ ...(ii)

Put $x = \sin t$ in (i) and (ii), we get

$$u=\sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2t}}\right)=\sec^{-1}\left(\frac{1}{\cos t}\right)=\sec^{-1}(\sec t)=t$$

and $v=\sin^{-1}(2\sin t\sqrt{1-\sin^2t})=\sin^{-1}(\sin 2t)=2t$

$$\therefore \frac{du}{dt}=1 \text{ and } \frac{dv}{dt}=2. \text{ So, } \frac{du}{dv}=\frac{du}{dt}\times\frac{dt}{dv}=\frac{1}{2}$$

61. Let $u=\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ ...(i)

Putting $x^2 = \cos 2\theta$ in (i), we get

$$u=\tan^{-1}\left[\frac{\sqrt{1+\cos2\theta}-\sqrt{1-\cos2\theta}}{\sqrt{1+\cos2\theta}+\sqrt{1-\cos2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]=\tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right]=\frac{\pi}{4}-\theta$$

$$=\frac{\pi}{4}-\frac{\cos^{-1}(x^2)}{2} \quad \left(\because x^2=\cos2\theta\Rightarrow\theta=\frac{\cos^{-1}x^2}{2}\right)$$

$$\Rightarrow \frac{du}{dx}=\frac{1}{2}\frac{1}{\sqrt{1-x^4}}2x=\frac{x}{\sqrt{1-x^4}} \quad ...(ii)$$

Let $v=\cos^{-1}(x^2) \Rightarrow \frac{dv}{dx}=\frac{-2x}{\sqrt{1-x^4}}$ ...(iii)

Dividing (ii) by (iii), we get $\frac{du}{dv}=-\frac{1}{2}$.

Key Points

- $1 + \cos 2\theta = 2\cos^2\theta$, $1 - \cos 2\theta = 2\sin^2\theta$.

62. We have, $x = a(2\theta - \sin 2\theta)$ ...(i)

and $y = a(1 - \cos 2\theta)$ ...(ii)

Differentiating (i) w.r.t. θ, we get

$$\frac{dx}{d\theta}=a(2-2\cos2\theta) \quad ...(iii)$$

Differentiating (ii) w.r.t. θ, we get

$$\frac{dy}{d\theta}=2a\sin2\theta \quad ...(iv)$$

$$\text{Now, } \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2a\sin2\theta}{a(2-2\cos2\theta)}=\frac{\sin2\theta}{1-\cos2\theta}$$

$$\therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\sin 2\left(\frac{\pi}{3}\right)}{1-\cos\frac{2\pi}{3}} = \frac{\sin\left(\pi-\frac{\pi}{3}\right)}{1-\cos\left(\pi-\frac{\pi}{3}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{3}\right)}{1+\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

Concept Applied

- If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}, f'(t) \neq 0$

63. $x = a\sin 2t\,(1+\cos 2t)$, $y = b\cos 2t\,(1-\cos 2t)$

Now, $\frac{dx}{dt} = 2a\cos 2t(1+\cos 2t) + a\sin 2t(-2\sin 2t)$

$= 2a\cos 2t + 2a[\cos^2 2t - \sin^2 2t]$

$= 2a\cos 2t + 2a\cos 4t$ $\quad (\because \cos^2(a) - \sin^2(a) = \cos(2a))$

Also, $\frac{dy}{dt} = -2b\sin 2t(1-\cos 2t) + b\cos 2t(2\sin 2t)$

$= -2b\sin 2t + 4b(\sin 2t\cos 2t)$

$= -2b\sin 2t + 2b\sin 4t$ $\quad (\because 2\sin(a)\cos(a) = \sin(2a))$

So, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(\sin 4t - \sin 2t)}{2a(\cos 4t + \cos 2t)}$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=\pi/4} = \frac{b}{a}\left[\frac{\sin\pi - \sin(\pi/2)}{\cos\pi + \cos(\pi/2)}\right] = \frac{b}{a}\left[\frac{0-1}{-1+0}\right] = \frac{b}{a}$$

$$\left(\frac{dy}{dx}\right)_{\text{at } t=\pi/3} = \frac{b}{a}\left[\frac{\sin(4\pi/3) - \sin(2\pi/3)}{\cos(4\pi/3) + \cos(2\pi/3)}\right]$$

$$= \frac{b}{a}\left[\frac{-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{-\frac{1}{2} - \frac{1}{2}}\right] = \frac{\sqrt{3}b}{a}$$

64. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\therefore u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) \Rightarrow u = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\therefore u = \frac{\theta}{2} \Rightarrow u = \frac{1}{2}\tan^{-1}x$$

$$\left(\because 2\sin^2\frac{x}{2} = 1-\cos x, 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x\right)$$

Differentiating w.r.t. x, we get $\frac{du}{dx} = \frac{1}{2(1+x^2)}$

Also, let $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \Rightarrow v = 2\tan^{-1}x$

Differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{1+x^2} \quad \therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}} \Rightarrow \frac{du}{dv} = \frac{1}{4}$$

65. We have $x = ae^t(\sin t + \cos t)$

$\Rightarrow \frac{dx}{dt} = ae^t(\sin t + \cos t) + ae^t(\cos t - \sin t) = 2ae^t\cos t$

and $y = ae^t(\sin t - \cos t)$

$\Rightarrow \frac{dy}{dt} = ae^t(\sin t - \cos t) + ae^t(\cos t + \sin t) = 2ae^t\sin t$

$\therefore$ L.H.S. $= \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2ae^t\sin t}{2ae^t\cos t} = \tan t$

Also, R.H.S. $= \frac{x+y}{x-y} = \frac{ae^t(\sin t + \cos t) + ae^t(\sin t - \cos t)}{ae^t(\sin t + \cos t) - ae^t(\sin t - \cos t)}$

$= \frac{2ae^t\sin t}{2ae^t\cos t} = \tan t = $ L.H.S.

66. Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

Put $x = \cos\theta$

$$\therefore u = \tan^{-1}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right] = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$= \tan^{-1}(\tan\theta) = \theta \Rightarrow \frac{du}{d\theta} = 1$

Also let,

$v = \cos^{-1}(2x\sqrt{1-x^2}) \Rightarrow v = \cos^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$

$= \cos^{-1}(2\cos\theta\sin\theta) = \cos^{-1}(\sin 2\theta)$

$= \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right) = \frac{\pi}{2} - 2\theta \Rightarrow \frac{dv}{d\theta} = -2$

Now $\frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = \frac{-1}{2}$

67. Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$\therefore u = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right) = \tan^{-1}(\tan\theta)$$

$\Rightarrow u = \theta \Rightarrow u = \sin^{-1}x$

Differentiating w.r.t. x, we get $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ ...(i)

Again, let $v = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta$

$\therefore v = \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow v = 2\sin^{-1}x$

Differentiating w.r.t. x, we get $\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$ ...(ii)

From (i) & (ii), we get $\frac{du}{dv} = \frac{1}{2}$

Concept Applied

- Power rule : $\frac{d}{dx}(x^n) = nx^{n-1}, n \neq 0$

68. We have, $x = ae^\theta(\sin\theta - \cos\theta)$
Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = ae^\theta(\sin\theta - \cos\theta) + ae^\theta(\cos\theta + \sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = ae^\theta(2\sin\theta) \quad \text{...(i)}$$

Also $y = ae^\theta(\sin\theta + \cos\theta)$
Differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = ae^\theta(\sin\theta + \cos\theta) + ae^\theta(\cos\theta - \sin\theta)$$

$$\Rightarrow \frac{dy}{d\theta} = ae^\theta(2\cos\theta) \quad \text{...(ii)}$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2ae^\theta\cos\theta}{2ae^\theta\sin\theta} = \cot\theta \quad \text{[From (i) \& (ii)]}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

69. Here, $x = \cos t(3 - 2\cos^2 t)$, $y = \sin t(3 - 2\sin^2 t)$

$$\Rightarrow \frac{dx}{dt} = -\sin t(3 - 2\cos^2 t) + \cos t[2\cdot 2\cos t\sin t]$$

$$= -3\sin t + 6\cos^2 t\sin t$$

and $\frac{dy}{dt} = \cos t(3 - 2\sin^2 t) + \sin t(-2\cdot 2\sin t\cos t)$

$$= 3\cos t - 6\sin^2 t\cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t - 6\sin^2 t\cos t}{-3\sin t + 6\cos^2 t\sin t} = \frac{3\cos t\cdot\cos 2t}{3\sin t\cdot\cos 2t} = \cot t$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

70. (d): We have, $x = A\cos 4t + B\sin 4t$
Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = A\cdot(-\sin 4t)\cdot 4 + B\cos 4t\cdot 4 \quad \text{...(i)}$$

Again differentiating both sides of (i) w.r.t. t, we get

$$\frac{d^2x}{dt^2} = -4A(\cos 4t)\cdot 4 + 4B(-\sin 4t)\cdot 4$$

$= -16A\cos 4t - 16B\sin 4t = -16(A\cos 4t + B\sin 4t) = -16x$

71. (b): Given, $y = e^{-x}$

$$\Rightarrow \frac{dy}{dx} = -e^{-x} \Rightarrow \frac{d^2y}{dx^2} = e^{-x} = y$$

72. (b): Given, $x = t^2 + 1$ and $y = 2at$

$$\Rightarrow \frac{dx}{dt} = 2t \Rightarrow \frac{dy}{dt} = 2a \therefore \frac{dy}{dx} = \frac{a}{t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a}{t^2}\cdot\frac{dt}{dx} = \frac{-a}{2t^3} \therefore \left(\frac{d^2y}{dx^2}\right)_{\text{at } t=a} = \frac{-a}{2a^3} = \frac{-1}{2a^2}$$

73. (c): We have, $y = \sin(2\sin^{-1}x)$

$$\Rightarrow y = \sin\left[\sin^{-1}\left(2x\sqrt{1-x^2}\right)\right]$$

$$\left[\because 2\sin^{-1}x = \sin^{-1}2x\sqrt{1-x^2}\right]$$

$$\Rightarrow y = 2x\sqrt{1-x^2} \quad \text{...(i)}$$

$$\Rightarrow y_1 = 2x\times\frac{-2x}{2\sqrt{1-x^2}} + 2\sqrt{1-x^2} = \frac{-4x^2+2}{\sqrt{1-x^2}} \quad \text{...(ii)}$$

$$\therefore y_2 = \frac{\sqrt{1-x^2}(-8x) - (-4x^2+2)\times\frac{-2x}{2\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{4x^3-6x}{(1-x^2)\sqrt{1-x^2}} \Rightarrow (1-x^2)y_2 = \frac{4x^3-6x}{\sqrt{1-x^2}}$$

Now, consider $xy_1 - 4y$

$$= \frac{-4x^3+2x}{\sqrt{1-x^2}} - 8x\sqrt{1-x^2} \quad \text{[Using (i) and (ii)]}$$

$$= \frac{4x^3-6x}{\sqrt{1-x^2}}$$

Thus, $(1 - x^2)y_2 = xy_1 - 4y$

74. (d): We have, $y = \log_e\left(\frac{x^2}{e^2}\right)$

$$\therefore \frac{dy}{dx} = \frac{e^2}{x^2}\cdot\frac{1}{e^2}\cdot 2x = \frac{2}{x} \Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{x^2}$$

75. Given, $x = at^2$, $y = 2at$

$$\therefore \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a \therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

$$\text{So, } \frac{d^2y}{dx^2} = -\frac{1}{t^2}\cdot\frac{dt}{dx} = -\frac{1}{t^2}\cdot\frac{1}{2at} = -\frac{1}{2at^3}$$

76. We have, $x = a\cos\theta$, $y = b\sin\theta$

$$\therefore \frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = b\cos\theta$$

$$\text{So, } \frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$\text{Now, } \frac{d^2y}{dx^2} = -\frac{b}{a}(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx}$$

$$= \frac{b}{a}\operatorname{cosec}^2\theta\left(-\frac{1}{a}\operatorname{cosec}\theta\right) = -\frac{b}{a^2}\operatorname{cosec}^3\theta$$

77. We have, $y = \tan x + \sec x$
Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \sec^2 x + \sec x\tan x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x} + \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \frac{1+\sin x}{\cos^2 x}$$

Now, again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\cos^2 x(\cos x) - (1+\sin x)(-2\cos x\sin x)}{(\cos^2 x)^2}$$

$$= \frac{\cos^3 x + (1+\sin x)\cdot 2\sin x\cdot\cos x}{(1-\sin^2 x)^2}$$

$$= \frac{\cos^3 x + 2\sin x\cos x + 2\sin^2 x\cos x}{(1-\sin x)^2(1+\sin x)^2}$$

$$=\frac{\cos x(\cos^2 x+\sin^2 x)+\sin^2 x\cos x+2\sin x\cos x}{(1-\sin x)^2(1+\sin x)^2}$$

$$=\frac{\cos x\{1+\sin^2 x+2\sin x\}}{(1-\sin x)^2(1+\sin x)^2}=\frac{\cos x\cdot(1+\sin x)^2}{(1-\sin x)^2(1+\sin x)^2}=\frac{\cos x}{(1-\sin x)^2}$$

Hence proved.

78. We have, $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$

$$\therefore \quad \frac{dx}{d\theta}=-a\sin\theta+b\cos\theta,\ \frac{dy}{d\theta}=a\cos\theta+b\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{a\cos\theta+b\sin\theta}{-a\sin\theta+b\cos\theta}=\frac{x}{-y}\quad \frac{d^2y}{dx^2}=\frac{-y\cdot 1-x\left(-\frac{dy}{dx}\right)}{(-y)^2}$$

$$\Rightarrow \quad y^2\frac{d^2y}{dx^2}=-y+x\frac{dy}{dx}\ \Rightarrow\ y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$$

79. Here $x = a\sec^3\theta$

$$\Rightarrow \quad \frac{dx}{d\theta}=a\cdot 3\sec^2\theta\cdot\sec\theta\tan\theta = 3a\sec^3\theta\tan\theta \text{ and } y=a\tan^3\theta$$

$$\Rightarrow \quad \frac{dy}{d\theta}=a\cdot 3\tan^2\theta\cdot\sec^2\theta$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta}=\frac{\tan\theta}{\sec\theta}=\sin\theta$$

On differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\cos\theta\cdot\frac{d\theta}{dx}=\frac{\cos\theta}{3a\sec^3\theta\tan\theta}=\frac{1}{3a}\cos^4\theta\cdot\cot\theta$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}}=\frac{1}{3a}\cos^4\frac{\pi}{4}\cdot\cot\frac{\pi}{4}=\frac{1}{3a}\cdot\left(\frac{1}{\sqrt{2}}\right)^4\cdot 1=\frac{1}{3a}\cdot\frac{1}{4}=\frac{1}{12a}$$

80. We have, $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow \quad \frac{dy}{dx}=-a\sin(\log x)\frac{1}{x}+b\cos(\log x)\cdot\frac{1}{x}$$

$$=\frac{-a\sin(\log x)+b\cos(\log x)}{x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=\frac{x\left[-a\cos(\log x)\cdot\frac{1}{x}-b\sin(\log x)\cdot\frac{1}{x}\right]-(-a\sin(\log x)+b\cos(\log x))}{x^2}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}=-y-\frac{xdy}{dx}\ \Rightarrow\ x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

81. We have, $y = (\sin^{-1}x)^2$ ...(i)

$$\Rightarrow \frac{dy}{dx}=2(\sin^{-1}x)\times\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{4(\sin^{-1}x)^2}{(1-x^2)}$$

[From (i)]

$$\Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{4y}{1-x^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(1-x^2)=4y \qquad ...(ii)$$

Again, differentiating (ii) w.r.t. x on both sides, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}(1-x^2)+\left(\frac{dy}{dx}\right)^2(-2x)=4\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0$$

Concept Applied

➲ Let $y = f(x)$, then $\frac{dy}{dx}=f'(x)$. If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right)=f''(x)$ or $\frac{d^2y}{dx^2}=f''(x)$

82. We have, $x = \sin t$ and $y = \sin pt$

$$\frac{dx}{dt}=\cos t \text{ and } \frac{dy}{dt}=p\cos pt\ \Rightarrow\ \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{p\cos pt}{\cos t}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{-p^2\sin pt\cos t+p\cos pt\sin t}{\cos^2 t}\times\frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=\frac{-p^2\sin pt\cos t+p\cos pt\sin t}{\cos^3 t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-\frac{p^2\sin pt\cos t}{\cos^3 t}+\frac{p\cos pt\sin t}{\cos^3 t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=\frac{-p^2y}{\cos^2 t}+\frac{x\frac{dy}{dx}}{\cos^2 t}\ \Rightarrow\ \cos^2 t\frac{d^2y}{dx^2}=-p^2y+x\frac{dy}{dx}$$

$$\Rightarrow \quad (1-\sin^2 t)\frac{d^2y}{dx^2}=-p^2y+x\frac{dy}{dx}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0$$

83. Here, $y = \sin(\sin x)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\cos(\sin x)\cdot\cos x$$

Again, differentiating w.r.t. x both sides, we get

$$\frac{d^2y}{dx^2} = -\sin(\sin x)\cdot\cos x\cdot\cos x + (-\sin x)\cos(\sin x)$$

$$= -\cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x)$$

Now, L.H.S. $=\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x$

$= -\cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x) + \tan x(\cos x\cdot\cos(\sin x)) + \cos^2 x\cdot\sin(\sin x)$

$= -\sin x\cdot\cos(\sin x) + \sin x\cdot\cos(\sin x)$

$= 0 =$ R.H.S.

84. Given $x^m y^n = (x + y)^{m+n}$

Taking log on both the sides, we get

$\log x^m + \log y^n = (m + n)\log(x + y)$

$\Rightarrow \quad m\log x + n\log y = (m + n)\log(x + y)$

Differentiating w.r.t. x, we get

$$m\cdot\frac{1}{x}+n\cdot\frac{1}{y}\cdot\frac{dy}{dx}=(m+n)\cdot\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{n}{y}-\frac{m+n}{x+y}\right)=\frac{m+n}{x+y}-\frac{m}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right)=\frac{mx+nx-mx-my}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right)=\frac{nx-my}{x(x+y)} \quad \therefore \quad \frac{dy}{dx}=\frac{y}{x}.$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{x\frac{dy}{dx}-y}{x^2}=\frac{x\left(\frac{y}{x}\right)-y}{x^2}=0 \therefore \frac{d^2y}{dx^2}=0$$

85. Given that $e^y.(x+1)=1$ ...(i)

Differentiating (i) w.r.t. x, we get

$$e^y\frac{d}{dx}(x+1)+(x+1)\frac{d}{dx}e^y=\frac{d}{dx}(1) \Rightarrow e^y+(x+1)e^y\frac{dy}{dx}=0$$

$$\Rightarrow e^y\left[1+(x+1)\frac{dy}{dx}\right]=0 \Rightarrow (x+1)\frac{dy}{dx}=-1$$

and $\frac{dy}{dx}=\frac{-1}{x+1}$ ...(ii)

or $\left(\frac{dy}{dx}\right)^2=\frac{1}{(x+1)^2}$ ...(iii)

Again, differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{1}{(x+1)^2} \Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2 \quad \text{[From (iii)]}$$

86. We have, $y=x^x \Rightarrow y=e^{x\log x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{x\log x}\left(x\times\frac{1}{x}+\log x\right)$$

$$\Rightarrow \frac{dy}{dx}=x^x(1+\log x) \Rightarrow \frac{dy}{dx}=y(1+\log x) \quad ...(1)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=(1+\log x)\cdot\frac{dy}{dx}+y\times\frac{1}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{y}\left(\frac{dy}{dx}\right)^2+\frac{y}{x} \quad \text{[From (1)]}$$

$$\Rightarrow \frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$$

87. We have, $y=2\cos(\log x)+3\sin(\log x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=-2\sin(\log x)\times\frac{1}{x}+3\cos(\log x)\times\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx}=-2\sin(\log x)+3\cos(\log x) \quad ...(1)$$

Again, differentiating w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-2\cos(\log x)\times\frac{1}{x}-3\sin(\log x)\times\frac{1}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-[2\cos(\log x)+3\sin(\log x)]$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y \Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

Answer Tips

- $\frac{d}{dx}(\log(x))=\frac{1}{x}$

88. Given, $x=a\cos\theta+b\sin\theta$, $y=a\sin\theta-b\cos\theta$

$\Rightarrow x^2=a^2\cos^2\theta+b^2\sin^2\theta+2ab\cos\theta\sin\theta$ ...(1)

and $y^2=a^2\sin^2\theta+b^2\cos^2\theta-2ab\sin\theta\cos\theta$ ...(2)

Adding (1) and (2), we get $x^2+y^2=a^2+b^2$

Differentiating w.r.t. x, we get

$$2x+2y\frac{dy}{dx}=0 \Rightarrow x+y\frac{dy}{dx}=0 \quad ...(3)$$

Again, differentiating w.r.t. x, we get

$$1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

Multiplying by y on both sides, we get

$$y^2\frac{d^2y}{dx^2}+\left(y\cdot\frac{dy}{dx}\right)\frac{dy}{dx}+y=0$$

$$\Rightarrow y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0 \quad \text{[From (3)]}$$

89. We have, $y=e^{m\sin^{-1}x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{m\sin^{-1}x}\left(\frac{m}{\sqrt{1-x^2}}\right)=\frac{my}{\sqrt{1-x^2}} \quad ...(1)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=m\left[\frac{\sqrt{1-x^2}\frac{dy}{dx}-\frac{y}{2\sqrt{1-x^2}}\cdot(-2x)}{(1-x^2)}\right]$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}=m\left[my+\frac{xy}{\sqrt{1-x^2}}\right] \quad \text{[From (1)]}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}=m\left[my+x\cdot\left(\frac{1}{m}\cdot\frac{dy}{dx}\right)\right]$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}=m^2y+x\frac{dy}{dx} \Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$$

90. We have, $y=(x+\sqrt{1+x^2})^n$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}=\frac{ny}{\sqrt{1+x^2}} \quad ...(1)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=n\left[\frac{\sqrt{1+x^2}\cdot\frac{dy}{dx}-\frac{2(xy)}{2\sqrt{1+x^2}}}{1+x^2}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n\left[\sqrt{1+x^2}\times\frac{ny}{\sqrt{1+x^2}}-\frac{xy}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-\frac{nxy}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-x\frac{dy}{dx} \quad \text{[From (1)]}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

91. Given $y = Ae^{mx} + Be^{nx}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{mx}\cdot m + Be^{nx}\cdot n \Rightarrow \frac{d^2y}{dx^2} = m^2Ae^{mx} + n^2Be^{nx}$$

Now, L.H.S. $= \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$= m^2Ae^{mx} + n^2Be^{nx} - (m+n)(mAe^{mx} + nBe^{nx}) + mn(Ae^{mx} + Be^{nx})$

$= Ae^{mx}[m^2 - (m+n)m + mn] + Be^{nx}[n^2 - (m+n)n + mn]$

$= Ae^{mx} \times 0 + Be^{nx} \times 0 = 0 =$ R.H.S.

92. Here, $x = a(\cos t + t\sin t)$

$$\Rightarrow \frac{dx}{dt} = a[-\sin t + 1\cdot\sin t + t\cos t] = at\cos t \quad ...(1)$$

and $y = a(\sin t - t\cos t)$

$$\Rightarrow \frac{dy}{dt} = a[\cos t - (1\cdot\cos t - t\sin t)] = a\,t\sin t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\,t\sin t}{a\,t\cos t} = \tan t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t\cdot\frac{dt}{dx} = \frac{\sec^2 t}{a\,t\cos t} \quad \text{[Using (1)]}$$

$$= \frac{1}{a}\cdot\frac{1}{t\cos^3 t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = \frac{1}{a}\cdot\frac{1}{\frac{\pi}{4}\cos^3\frac{\pi}{4}} = \frac{4}{\pi a}\cdot(\sqrt{2})^3 = \frac{8\sqrt{2}}{\pi a}$$

93. Here, $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$

$$\Rightarrow \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}\cdot\frac{1}{2}\sec^2\frac{t}{2}\right)$$

$$= a\left(-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\cdot\frac{1}{2\cos^2\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\frac{(-\sin^2 t + 1)}{\sin t} = \frac{a\cos^2 t}{\sin t}$$

Also, $y = a\sin t \Rightarrow \frac{dy}{dt} = a\cos t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = a\cos t\cdot\frac{\sin t}{a\cos^2 t} = \tan t$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t\cdot\frac{dt}{dx} = \frac{\sec^2 t}{a\cos^2 t/\sin t} = \frac{\sin t}{a\cos^4 t}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{a\cos^4\frac{\pi}{3}} = \frac{\sqrt{3}/2}{a(1/2)^4} = \frac{8\sqrt{3}}{a}$$

CBSE Sample Questions

1. (c): Given, the function f is continuous at $x = 0$.

$\therefore \lim_{x\to 0} f(x) = f(0)$.

Now, $\lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}\frac{1-\cos 4x}{8x^2} = \lim_{x\to 0}\frac{2\sin^2 2x}{8x^2} = \lim_{x\to 0}\frac{\sin^2 2x}{4x^2}$$

$$= \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1 \quad (1)$$

Also, $f(0) = k$

Hence, $k = 1$

2. (b): We have, $f(x) = \begin{cases}\frac{1-\cos kx}{x\sin x}, & x \neq 0\\ \frac{1}{2}, & x = 0\end{cases}$

$\because$ $f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x\to 0}\frac{1-\cos kx}{x\sin x} = \frac{1}{2} \Rightarrow \lim_{x\to 0}\frac{2\sin^2\frac{kx}{2}}{x^2\frac{\sin x}{x}} = \frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} 2\cdot\frac{k^2}{4}\left\{\frac{\sin\left(\frac{kx}{2}\right)}{\frac{kx}{2}}\right\}^2\frac{1}{\frac{(\sin x)}{x}} = \frac{1}{2}$$

$$\Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1$$

But $k < 0$ $\therefore$ $k = -1$ (1)

3. (a): We have, $f(x) = \begin{cases}\frac{x}{|x|}, & x < 0\\ -1, & x \geq 0\end{cases}$

$$\Rightarrow f(x) = \begin{cases}\frac{x}{-x} = -1, & x < 0\\ -1, & x \geq 0\end{cases}$$

$\Rightarrow$ $f(x) = -1\ \forall\ x \in R$

$\Rightarrow$ $f(x)$ is continuous $\forall\ x \in R$ as it is a constant function. (1)

4. We have, $\lim_{x\to 0}\frac{1-\cos kx}{x\sin x} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{kx}{2}\right)}{x\sin x}$

$$= \frac{\lim_{x\to 0}\frac{2\sin^2\left(\frac{kx}{2}\right)}{\left(\frac{kx}{2}\right)^2}\times\left(\frac{k}{2}\right)^2}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{2\times 1\times\frac{k^2}{4}}{1} = \frac{k^2}{2} \quad (1\tfrac{1}{2})$$

$\because$ $f(x)$ is continuous at $x = 0$.

$\therefore \lim_{x\to 0} f(x) = f(0)$

$$\Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k^2 = 1 \Rightarrow k = \pm 1 \quad (1/2)$$

5. We have, $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$ ...(i)

Let $\sin^{-1}x = A$ and $\sin^{-1}y = B$.

Then, $x = \sin A$ and $y = \sin B$

From (i) $\sin B\cos A + \sin A\cos B = 1$

$\Rightarrow$ $\sin(A + B) = 1$ (1)

$\Rightarrow \quad A+B=\sin^{-1}1=\frac{\pi}{2}$

$\Rightarrow \sin^{-1}x+\sin^{-1}y=\frac{\pi}{2}$

Differentiating w.r.t. x, we obtain $\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$ (1)

6. We have, $f(x)=[x], 0<x<2$.

R.H.D.(at $x=1$) $=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{[1+h]-[1]}{h}$

$=\lim_{h\to 0}\frac{(1-1)}{h}=0$ (1)

L.H.D.(at $x=1$) $=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{[1-h]-[1]}{-h}$

$=\lim_{h\to 0}\frac{0-1}{-h}=\lim_{h\to 0}\frac{1}{h}=\infty$ (1)

Since, R.H.D. $\neq$ L.H.D.

Therefore $f(x)$ is not differentiable at $x=1$. (1)

7. (c): We have, $e^x+e^y=e^{x+y}$

$\Rightarrow e^{-y}+e^{-x}=1$

Differentiating w.r.t. x, we get

$-e^{-y}\frac{dy}{dx}-e^{-x}=0\Rightarrow\frac{dy}{dx}=-e^{y-x}$ (1)

8. (d): We have, $y=\log(\cos e^x)$

Differentiating both sides w.r.t. x, we get

$\frac{dy}{dx}=\frac{1}{\cos e^x}\cdot(-\sin e^x)\cdot e^x$

$\Rightarrow \frac{dy}{dx}=-e^x\tan e^x$ (1)

9. We have, $y=e^{x\sin^2x}+(\sin x)^x \Rightarrow y=u+v$,

where $u=e^{x\sin^2x}$ and $v=(\sin x)^x$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ... (i)

Now, consider $u=e^{x\sin^2x}$

Differentiating both sides with respect to x, we get

$\frac{du}{dx}=e^{x\sin^2x}[x\cdot(2\sin x\cos x)+\sin^2x]$ (1)

$=e^{x\sin^2x}[x(\sin 2x)+\sin^2x]$ ... (ii)

Also, $v=(\sin x)^x$

$\Rightarrow \log v=x\log(\sin x)$

Differentiating both sides with respect to x, we get

$\frac{1}{v}\frac{dv}{dx}=x\cdot\frac{1}{\sin x}\cdot\cos x+\log(\sin x)$

$\Rightarrow \frac{dv}{dx}=(\sin x)^x[x\cot x+\log(\sin x)]$ ... (iii) (1/2)

From (i), (ii) and (iii), we get

$\frac{dy}{dx}=e^{x\sin^2x}[x\sin 2x+\sin^2x]+(\sin x)^x[x\cot x+\log(\sin x)]$ (1/2)

10. (a): Let $u=\sin^{-1}(2x\sqrt{1-x^2})$ ...(i)

and $v=\sin^{-1}x, \frac{1}{\sqrt{2}}<x<1$

$\Rightarrow \sin v=x$ ...(ii)

From (i) and (ii), we get

$u=\sin^{-1}(2\sin v\cos v)=\sin^{-1}(\sin 2v)$

$\Rightarrow u=2v$

Differentiating with respect to v both sides, we get

$\frac{du}{dv}=2$ (1)

11. (a): We have, $y=5\cos x-3\sin x$

$\Rightarrow \frac{dy}{dx}=-5\sin x-3\cos x$

$\Rightarrow \frac{d^2y}{dx^2}=-5\cos x+3\sin x=-y$ (1)

12. (a): We have, $x=a\sec\theta$

$\Rightarrow \frac{dx}{d\theta}=a\tan\theta\sec\theta$ and $y=b\tan\theta\Rightarrow\frac{dy}{d\theta}=b\sec^2\theta$

$\therefore \frac{dy}{dx}=\frac{b\sec^2\theta}{a\tan\theta\sec\theta}=\frac{b}{a}\text{cosec}\theta$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{-b}{a}\text{cosec}\theta\cot\theta\frac{d\theta}{dx}$

$=\frac{-b}{a}\text{cosec}\theta\cot\theta\frac{1}{a\tan\theta\sec\theta}=\frac{-b}{a^2}\cot^3\theta$

$\therefore \left[\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{6}}=\frac{-3\sqrt{3}b}{a^2}$ (1)

13. We have, $y=b\tan\theta\Rightarrow\frac{dy}{d\theta}=b\sec^2\theta$ ... (i)

and $x=a\sec\theta\Rightarrow\frac{dx}{d\theta}=a\sec\theta\tan\theta$ ... (ii)

$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b}{a}\text{cosec}\theta$ (1½)

Differentiating both sides with respect to x, we get

$\frac{d^2y}{dx^2}=\frac{-b}{a}\text{cosec}\theta\cot\theta\times\frac{d\theta}{dx}$

$=-\frac{b}{a}\text{cosec}\theta\cot\theta\times\frac{1}{a\sec\theta\tan\theta}$ [using (ii)] (1)

$=-\frac{b}{a^2}\cot^3\theta$

$\therefore \left[\frac{d^2y}{dx^2}\right]_{\theta=\pi/6}=\frac{-b}{a^2}\left[\cot\frac{\pi}{6}\right]^3=\frac{-b}{a^2}(\sqrt{3})^3=-\frac{3\sqrt{3}b}{a^2}$ (1/2)